Pergamon

Computers Math. Applic. Vol. 36, No. 8, pp. 59–65, 1998

Printed in Great Britain
0898-1221/98 $19.00 + 0.00

S0898-1221(98)00183-7

An Effective Numerical Method to Compute the Moments of the Completion Time of Markov Reward Models

M. TELEK AND A. PFENING
Department of Telecommunications, Technical University of Budapest
1521 Budapest, Hungary

G. FODOR
Department of Telecommunications and Telematics
Technical University of Budapest, 1521 Budapest, Hungary

(Received March 1998; accepted April 1998)

Abstract—Analysis of Markov Reward Models (MRM) with preemptive resume (prs) policy results in a double transform expression, whose solution is based on the inverse transformations both in time and reward variable domain. We present a symbolic expression of moments of the completion time, from which a computationally effective recursive numerical method can be obtained. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords—Markov reward models, Preemptive resume policy, Completion time.

1. INTRODUCTION

Stochastic Reward Models (SRM) are composed by a discrete state stochastic process (referred to as structure-state process) and an associated continuous variable (referred to as reward variable) whose evolution is controlled by the stochastic process. In the considered class of models, the accumulated reward increases at a rate associated to the current state of the structure-state process. The rate associated to each structure-state is called the *reward rate* [1].

Let the *structure-state process* $Z(t)$ $(t \geq 0)$ be a (right continuous) stochastic process defined over a discrete and finite state space Ω of cardinality n. Let f be a nonnegative real-valued function defined as

$$f[Z(t)] = r_i \geq 0, \qquad \text{if} \quad Z(t) = i, \tag{1}$$

$f[Z(t)]$ represents the instantaneous reward rate associated to state i.

DEFINITION 1. *The accumulated reward $B(t)$ is a random variable which represents the accumulation of reward in time*

$$B(t) = \int_0^t f[Z(\tau)]\, d\tau = \int_0^t r_{Z(\tau)}\, d\tau.$$

$B(t)$ is a stochastic process that depends on $Z(u)$ for $0 \leq u \leq t$. According to Definition 1, this paper restricts the attention to the class of models in which no state transition can entail to

This work was partially supported by Hungarian Grant OTKA T-16637.

Typeset by $\mathcal{AMS}$-TeX

a loss of the accumulated reward. An SRM of this kind is called *preemptive resume* (prs) model. The distribution of the accumulated reward is defined as

$$B(t, w) = \Pr\{B(t) \le w\}.$$

The complementary question concerning the reward accumulation of SRMs is the time needed to complete a given (possibly random) work requirement (i.e., the time to accumulate the required amount of reward).

DEFINITION 2. *The completion time C is the random variable representing the time to accumulate a reward requirement equal to a random variable W*

$$C = \min[t \ge 0 : B(t) = W].$$

C is the time instant at which the work accumulated by the system reaches the value W for the first time. Assume, in general, that W is a random variable with distribution $G(w)$ with support on $(0, \infty)$, and that W is independent from $Z(t)$. The degenerate case, in which W is deterministic and the distribution $G(w)$ becomes the unit step function $U(w - w_d)$, can be considered as well. For a given sample of $W = w$, the completion time $C(w)$ and its cumulative distribution function $C(t, w)$ are defined as

$$C(w) = \min[t \ge 0 : B(t) = w], \qquad C(t, w) = \Pr\{C(w) \le t\}. \tag{2}$$

The completion time C is characterized by the following distribution:

$$\hat{C}(t) = \Pr\{C \le t\} = \int_0^\infty C(t, w)\, dG(w). \tag{3}$$

The distribution of the completion time of a prs SRM is closely related to the distribution of the accumulated reward by means of the following relation:

$$B(t, w) = \Pr\{B(t) \le w\} = \Pr\{C(w) \ge t\} = 1 - C(t, w). \tag{4}$$

For the purposes of the subsequent analysis below, we define the following matrix functions $\mathbf{P}(t, w) = \{P_{ij}(t, w)\}$ and $\mathbf{F}(t, w) = \{F_{ij}(t, w)\}$ as

$$P_{ij}(t, w) = \Pr\{Z(t) = j,\ B(t) \le w \mid Z(0) = i\}, \tag{5}$$

$$F_{ij}(t, w) = \Pr\{Z(C(w)) = j,\ C(w) \le t \mid Z(0) = i\}, \tag{6}$$

- where $P_{ij}(t, w)$ is the joint distribution of the accumulated reward and the structure state at time t supposed that the initial state of the structure state process is i,
- $F_{ij}(t, w)$ is the joint distribution of the completion time and the structure state at completion supposed that the initial state of the structure state process is i.

From (5) and (6), it follows for any t and i that $\sum_{j \in \Omega} [P_{ij}(t, w) + F_{ij}(t, w)] = 1$. By these definitions,

$$B(t, w) = \underline{P}(0)\mathbf{P}(t, w)\underline{h}^T \quad \text{and} \quad C(t, w) = \underline{P}(0)\mathbf{F}(t, w)\underline{h}^T,$$

where $\underline{P}(0)$ is the row vector of the initial probabilities, and $\underline{h}^T$ is the column vector with all the entries equal to 1.

Given that $G(w)$ is the cumulative distribution function of the random work requirement W, and from equation (3), the distribution of the completion time can be written as

$$\hat{C}(t) = \int_{w=0}^\infty \left[\sum_{i \in \Omega} \sum_{j \in \Omega} P_i(0) F_{ij}(t, w) \right] dG(w) = \int_{w=0}^\infty \underline{P}(0)\mathbf{F}(t, w)\underline{h}^T\, dG(w). \tag{7}$$

2. MARKOV REWARD MODELS

DEFINITION 3. *The subclass of SRMs in which the structure state process ($Z(t)$) is a Continuous Time Markov Chain (CTMC) is called Markov Reward Model (MRM).*

The introduced matrix functions of a MRM can be described in double transform domain based on the infinitesimal generator ($\mathbf{A}$) of the structure state process and the reward rates. The infinitesimal generator $\mathbf{A}$ is assumed to be irreducible, that is all states of the structure state process are transient. Detailed derivations presented in [2] results,

$$F_{ij}^{\sim *}(s,v) = \delta_{ij}\frac{r_i}{s+vr_i-a_{ii}} + \sum_{k\in R,\ k\neq i} \frac{a_{ik}}{s+vr_i-a_{ii}} F_{kj}^{\sim *}(s,v), \tag{8}$$

$$P_{ij}^{\sim *}(s,v) = \delta_{ij}\frac{s}{v(s+vr_i-a_{ii})} + \sum_{k\in R,\ k\neq i} \frac{a_{ik}}{s+vr_i-a_{ii}} P_{kj}^{\sim *}(s,v), \tag{9}$$

where $\sim$ denotes the Laplace-Stieltjes transform with respect to $t(\rightarrow s)$, $*$ denotes the Laplace transform with respect to $w(\rightarrow v)$, $\mathbf{I}$ is the identity matrix and $\mathbf{R}$ is the diagonal matrix of the reward rates (r_i), and δ_{ij} is the Kronecker delta. The final expressions take the following matrix forms:

$$\mathbf{F}^{\sim *}(s,v) = (s\mathbf{I} + v\mathbf{R} - \mathbf{A})^{-1}\mathbf{R}, \tag{10}$$

$$\mathbf{P}^{\sim *}(s,v) = \frac{s}{v}(s\mathbf{I} + v\mathbf{R} - \mathbf{A})^{-1}. \tag{11}$$

The dimensions of $\mathbf{I}$, $\mathbf{R}$, $\mathbf{A}$, $\mathbf{F}$, and $\mathbf{P}$ are $(n \times n)$.

Starting from equations (10),(11), the evaluation of the reward measures of a MRM requires the following steps.

(1) Symbolic evaluation of the entries of the $\mathbf{P}^{*\sim}(s,v)$ and $\mathbf{F}^{*\sim}(s,v)$ matrices in the double transform domain according to (10) and (11), which requires a symbolic inversion of an $n \times n$ size matrix.
(2) Symbolic inverse Laplace-Stieltjes transformation of $\mathbf{P}^{*\sim}(s,v)$ and/or $\mathbf{F}^{*\sim}(s,v)$ with respect to s.
(3) Numerical inverse Laplace transformation with respect to v.
(4) Unconditioning of the result by a numerical integration according to the distribution of the work requirement defined by (7).

However, this way of the analysis contains some computationally intensive steps, and the whole procedure can be applied to very small scale problems (less than 6–8 states) only.

3. COMPLETION TIME ANALYSIS OF MRMS

According to the associated reward rates, the states of MRMs can be divided into two parts, namely S and $S^c = \Omega - S$, where S contains the states with positive reward rates and S^c with zero reward rates, i.e., $\forall\, i \in S$, $r_i > 0$ and $\forall\, i \in S^c$, $r_i = 0$. Suppose that S contains m states out of n. Thus, we can renumber the states in Ω in a way that the states numbered $1, 2, \dots, m$ belong to S and the states numbered $m+1, m+2, \dots, n$ belong to S^c. By this ordering of the states, $\mathbf{A}$ can be partitioned into the following form: $\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix}$, where $\mathbf{A}_1$ describes the transitions inside S, $\mathbf{A}_2$ contains the intensity of the transitions from S to S^c, $\mathbf{A}_3$ the transitions from S^c to S, and $\mathbf{A}_4$ the transitions inside S^c. If there is no absorbing state group in S^c, i.e., the completion time of a finite work requirement w is finite with probability 1, then $\mathbf{A}_4^{-1}$ exists. By the renumbering of states, the diagonal matrix of the reward rates has the form $\mathbf{R} = \begin{bmatrix} \mathbf{R}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, where $\mathbf{R}_1 = \mathrm{Diag}_{i\in S}\langle r_i \rangle$ is the diagonal matrix of the reward rates in S with cardinality $m \times m$.

3.1. Moments of the Completion Time of MRMs

In this section, we calculate the moments of the completion time using the Laplace-Stieltjes transform, and we propose a recursive method to calculate the moments in a computationally effective way. We make use of the idea proposed by Iyer *et al.* for the analysis of the accumulated reward [3]. The n^{th} moment of the completion time of w amount of work is defined by

$$M_{(n)}(w) = E\{C(w)^n\} = \int_{t=0}^{\infty} t^n \, dC(t, w).$$

THEOREM 1. *The n^{th} moment of the completion time of an MRM with work requirement w is*

$$M_{(n)}(w) = n! \underline{P}(0) \mathrm{LT}^{-1} \left[(\mathbf{R}v - \mathbf{A})^{-(n+1)} \mathbf{R} \right] \underline{h}^T, \tag{12}$$

where LT^{-1} means the inverse Laplace transformation with respect to v.

PROOF. The moments can be calculated using Laplace-Stieltjes transform of the completion time and substituting equation (10),

$$\begin{aligned} M_{(n)}(w) &= (-1)^n \frac{\partial^n \mathrm{LT}^{-1}[C^{\sim *}(s,v)]}{\partial s^n} \bigg|_{s=0} \\ &= (-1)^n \frac{\partial^n \mathrm{LT}^{-1}\left[\underline{P}(0)\mathbf{F}^{\sim *}(s,v)\underline{h}^T\right]}{\partial s^n} \bigg|_{s=0} \\ &= (-1)^n \underline{P}(0) \frac{\partial^n \mathrm{LT}^{-1}[\mathbf{F}^{\sim *}(s,v)]}{\partial s^n} \underline{h}^T \bigg|_{s=0} \\ &= (-1)^n \underline{P}(0) \frac{\partial^n \mathrm{LT}^{-1}[(s\mathbf{I} + v\mathbf{R} - \mathbf{A})^{-1}\mathbf{R}]}{\partial s^n} \bigg|_{s=0} \underline{h}^T. \end{aligned} \tag{13}$$

We assumed in the above formula that the order of the inversion and the derivation can be changed

$$M_{(n)}(w) = (-1)^n \underline{P}(0) \mathrm{LT}^{-1} \left[\frac{\partial^n (s\mathbf{I} + v\mathbf{R} - \mathbf{A})^{-1}\mathbf{R}}{\partial s^n} \bigg|_{s=0} \right] \underline{h}^T.$$

The derivation can be accomplished using Leibniz's rule, and setting the value of s to 0,

$$M_{(n)}(w) = n! \underline{P}(0) \mathrm{LT}^{-1} \left[(v\mathbf{R} - \mathbf{A})^{-(n+1)} \mathbf{R} \right] \underline{h}^T. \qquad \blacksquare$$

3.2. Analysis of the Mean Completion Time of MRMs

Because of the inverse Laplace transformation and matrix inversion contained in equation (12), the calculation of the moments is a computationally intensive task. Begain *et al.* [4] proposed an effective method to calculate the first moment, i.e., the mean value of the completion time of on-off reward models. Here, we generalize that results for the mean completion time of MRMs with general reward structure.

THEOREM 2. *The expected time while a MRM with general reward rates completes w amount of work is*

$$E\{C(w)\} = \underline{P}(0) \begin{bmatrix} \mathbf{L}(w) & -\mathbf{L}(w)\mathbf{A}_2\mathbf{A}_4^{-1} \\ -\mathbf{A}_4^{-1}\mathbf{A}_3\mathbf{L}(w) & -\mathbf{A}_4^{-1} + \mathbf{A}_4^{-1}\mathbf{A}_3\mathbf{L}(w)\mathbf{A}_2\mathbf{A}_4^{-1} \end{bmatrix} \underline{h}^T, \tag{14}$$

where

$$\mathbf{L}(w) = \int_0^{\infty} e^{u\mathbf{R}_1^{-1}\beta} \, du \mathbf{R}_1^{-1} \quad \text{and} \quad \beta = \mathbf{A}_1 - \mathbf{A}_2\mathbf{A}_4^{-1}\mathbf{A}_3.$$

PROOF.

$$
\begin{aligned}
E\{C(w)\} &= \int_{t=0}^{\infty} (1 - C(t,w))\, dt = \int_{t=0}^{\infty} B(t,w)\, dt \\
&= \lim_{s\to 0} \frac{1}{s} B^{\sim}(s,w) = \lim_{s\to 0} \underline{P}(0)^T \mathbf{P}^{\sim}(s,w)\underline{h}^T \\
&= \underline{P}(0) \mathrm{LT}^{-1}\left[\frac{1}{v}(v\mathbf{R} - \mathbf{A})^{-1}\right] \underline{h}^T.
\end{aligned} \tag{15}
$$

Let us consider the term $\mathrm{LT}^{-1}\left[(1/v)(v\mathbf{R} - \mathbf{A})^{-1}\right]$ separately using the partitioned form $\mathbf{R} = \left[\begin{smallmatrix} \mathbf{R}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{smallmatrix}\right]$. Using the inverse of a partitioned matrix, the inverse Laplace transform is as follows:

$$
\mathrm{LT}^{-1}\left[\frac{1}{v}(v\mathbf{R} - \mathbf{A})^{-1}\right] = \mathrm{LT}^{-1}\left\{\frac{1}{v}\begin{bmatrix} v\mathbf{R}_1 - \mathbf{A}_1 & -\mathbf{A}_2 \\ -\mathbf{A}_3 & -\mathbf{A}_4 \end{bmatrix}^{-1}\right\}
$$

$$
= \mathrm{LT}^{-1}\left\{\frac{1}{v}\begin{bmatrix} (v\mathbf{R}_1 - \beta)^{-1} & -(v\mathbf{R}_1 - \beta)^{-1}\mathbf{A}_2\mathbf{A}_4^{-1} \\ -\mathbf{A}_4^{-1}\mathbf{A}_3(v\mathbf{R}_1 - \beta)^{-1} & \mathbf{A}_4^{-1} + \mathbf{A}_4^{-1}\mathbf{A}_3(v\mathbf{R}_1 - \beta)^{-1}\mathbf{A}_2\mathbf{A}_4^{-1} \end{bmatrix}\right\} \tag{16}
$$

$$
= \mathrm{LT}^{-1}\left\{\frac{1}{v}\begin{bmatrix} \left(v\mathbf{I}_1 - \mathbf{R}_1^{-1}\beta\right)^{-1}\mathbf{R}_1^{-1} & -\left(v\mathbf{I}_1 - \mathbf{R}_1^{-1}\beta\right)^{-1}\mathbf{R}_1^{-1}\mathbf{A}_2\mathbf{A}_4^{-1} \\ -\mathbf{A}_4^{-1}\mathbf{A}_3\left(v\mathbf{I}_1 - \mathbf{R}_1^{-1}\beta\right)^{-1}\mathbf{R}_1^{-1} & \mathbf{A}_4^{-1} + \mathbf{A}_4^{-1}\mathbf{A}_3\left(v\mathbf{I}_1 - \mathbf{R}_1^{-1}\beta\right)^{-1}\mathbf{R}_1^{-1}\mathbf{A}_2\mathbf{A}_4^{-1} \end{bmatrix}\right\}
$$

$$
= \begin{bmatrix} \mathbf{L}(w) & -\mathbf{L}(w)\mathbf{A}_2\mathbf{A}_4^{-1} \\ -\mathbf{A}_4^{-1}\mathbf{A}_3\mathbf{L}(w) & \mathbf{A}_4^{-1} + \mathbf{A}_4^{-1}\mathbf{A}_3\mathbf{L}(w)\mathbf{A}_2\mathbf{A}_4^{-1} \end{bmatrix}.
$$

From (16), the theorem follows. ∎

An intuitive understanding of Theorem 2 is possible based on the interpretation of matrix β. Define $Z'(t')$ a continuous time Markov chain over S based on the original structure state process $Z(t) \in \Omega$ as follows:

$$
Z'(t') = Z(t), \qquad \begin{aligned} \frac{dt'}{dt} &= 1, \quad \text{if } Z(t) \in S, \\ \frac{dt'}{dt} &= 0, \quad \text{if } Z(t) \in \Omega - S, \end{aligned}
$$

i.e., $Z'(t')$ takes the same state as $Z(t)$ when $Z(t) \in S$ and the clock t' is switched on (off) when $Z(t) \in S$ ($Z(t) \in \Omega - S$). β is the infinitesimal generator of CTMC $Z'(t')$ over S (with the usual properties: $\forall i,j \in S$, $\beta_{ij}\,|_{i\neq j} \geq 0$ and $\sum_{j\in S}\beta_{ij} = 0$). The multiplication with $\mathbf{R}_1^{-1}$ stands for scaling and rescaling the time providing a constant reward increment rate as proposed by Beaudry [5]. $Z'(t')$ is the stochastic process which characterize the reward accumulation as captured by $\mathbf{L}(w)$. The submatrices in (14) accounts for the time $Z(t)$ spends out of S.

3.3. A Recursive Analysis of Higher Moments

Here, we propose a recursive method to calculate the higher moments. First, we introduce some notation. Let $M_{ij(n)}(w)$ be the n^{th} moment of the completion time assuming that the process was started in state i, the work requirement was completed in state j, and the work requirement was w. Let $\mathbf{M}_{(n)}(w)$ be a matrix with entries $M_{ij(n)}(w)$, and $\mathbf{M}^*_{(n)}(v)$ be the Laplace transform of $\mathbf{M}_{(n)}(w)$. Let

$$
\mathbf{F}^{\sim *(n)}(0,v) = \frac{\partial^n \mathbf{F}^{\sim *}(s,v)}{\partial s^n}\bigg|_{s=0}.
$$

THEOREM 3. *The n^{th} moment ($n \geq 2$) of the completion time of an on-off MRM with binary reward rates and work requirement w can be obtained as*

$$
\begin{aligned}
M_{(n)}(w) &= \underline{P}(0)\mathbf{M}_{(n)}(w)\underline{h}^T \\
&= n\underline{P}(0)\int_{y=0}^{w} \Theta(w-y)\mathbf{M}_{(n-1)}(y)\underline{h}^T\, dy + n\underline{P}(0)\hat{\mathbf{A}}\mathbf{M}_{(n-1)}(w)\underline{h}^T,
\end{aligned} \tag{17}
$$

where

$$\Theta(w) = \begin{bmatrix} e^{w\mathbf{R}_1^{-1}\beta}\mathbf{R}_1^{-1} & -e^{w\mathbf{R}_1^{-1}\beta}\mathbf{R}_1^{-1}\mathbf{A}_2\mathbf{A}_4^{-1} \\ -\mathbf{A}_4^{-1}\mathbf{A}_3 e^{w\mathbf{R}_1^{-1}\beta}\mathbf{R}_1^{-1} & \mathbf{A}_4^{-1}\mathbf{A}_3 e^{w\mathbf{R}_1^{-1}\beta}\mathbf{R}_1^{-1}\mathbf{A}_2\mathbf{A}_4^{-1} \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{A}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{A}_4^{-1} \end{bmatrix}.$$

PROOF. From equation (10),

$$(s\mathbf{I} + v\mathbf{R} - \mathbf{A})\mathbf{F}^{\sim *}(s, v) = \mathbf{R}. \tag{18}$$

Using Leibniz's rule, the differentiation of equation (18) $n+1$ times with respect to s and setting $s = 0$ yields

$$\mathbf{F}^{\sim *(n+1)}(0, v) = -(n+1)(\mathbf{R}v - \mathbf{A})^{-1}\mathbf{F}^{\sim *(n)}(0, v). \tag{19}$$

Because $\mathbf{M}^*_{(n)}(v) = (-1)^n \mathbf{F}^{\sim *}(0, v)$ according to equation (13), equation (19) can be rewritten as

$$\mathbf{M}^*_{(n+1)}(v) = (n+1)(\mathbf{R}v - \mathbf{A})^{-1}\mathbf{M}^*_{(n)}(v). \tag{20}$$

Let us consider the term $\mathrm{LT}^{-1}[(v\mathbf{R} - \mathbf{A})^{-1}]$ separately. By the partitioned form of $\mathbf{R}$ and $\mathbf{A}$, the inverse Laplace transform satisfies the following equation:

$$\begin{aligned}
\mathrm{LT}^{-1}\left[(v\mathbf{R} - \mathbf{A})^{-1}\right] &= \mathrm{LT}^{-1}\left\{ \begin{bmatrix} v\mathbf{R}_1 - \mathbf{A}_1 & -\mathbf{A}_2 \\ -\mathbf{A}_3 & -\mathbf{A}_4 \end{bmatrix}^{-1} \right\} \\
&= \mathrm{LT}^{-1} \begin{bmatrix} (v\mathbf{R}_1 - \beta)^{-1} & (v\mathbf{R}_1 - \beta)^{-1}\mathbf{A}_2\mathbf{A}_4^{-1} \\ -\mathbf{A}_4^{-1}\mathbf{A}_3(v\mathbf{R}_1 - \beta)^{-1} & \mathbf{A}_4^{-1} + \mathbf{A}_4^{-1}\mathbf{A}_3(v\mathbf{R}_1 - \beta)^{-1}\mathbf{A}_2\mathbf{A}_4^{-1} \end{bmatrix} \\
&= \mathrm{LT}^{-1} \begin{bmatrix} (v\mathbf{I}_1 - \mathbf{R}_1^{-1}\beta)^{-1}\mathbf{R}_1^{-1} & (v\mathbf{I}_1 - \mathbf{R}_1^{-1}\beta)^{-1}\mathbf{R}_1^{-1}\mathbf{A}_2\mathbf{A}_4^{-1} \\ -\mathbf{A}_4^{-1}\mathbf{A}_3(v\mathbf{I}_1 - \mathbf{R}_1^{-1}\beta)^{-1}\mathbf{R}_1^{-1} & \mathbf{A}_4^{-1} + \mathbf{A}_4^{-1}\mathbf{A}_3(v\mathbf{I}_1 - \mathbf{R}_1^{-1}\beta)^{-1}\mathbf{R}_1^{-1}\mathbf{A}_2\mathbf{A}_4^{-1} \end{bmatrix} \\
&= \begin{bmatrix} e^{w\mathbf{R}_1^{-1}\beta}\mathbf{R}_1^{-1} & e^{w\mathbf{R}_1^{-1}\beta}\mathbf{R}_1^{-1}\mathbf{A}_2\mathbf{A}_4^{-1} \\ -\mathbf{A}_4^{-1}\mathbf{A}_3 e^{w\mathbf{R}_1^{-1}\beta}\mathbf{R}_1^{-1} & \mathbf{A}_4^{-1}\delta(w) + \mathbf{A}_4^{-1}\mathbf{A}_3 e^{w\mathbf{R}_1^{-1}\beta}\mathbf{R}_1^{-1}\mathbf{A}_2\mathbf{A}_4^{-1} \end{bmatrix}.
\end{aligned} \tag{21}$$

Hence, $\mathrm{LT}^{-1}[(v\mathbf{R} - \mathbf{A})^{-1}] = \Theta(w) + \hat{\mathbf{A}}\delta(w)$, where $\delta(w)$ denotes the Dirac delta function, the inversion and the integration yields the theorem. ∎

To apply the result of Theorem 3 for the evaluation of the first moment, we shall define in accordance with equation (13),

$$\begin{aligned}
&M_{(0)}(w) = \mathrm{LT}^{-1}\left[C^{\sim *}(0, v)\right] = \mathrm{LT}^{-1}\left[\underline{P}(0)\mathbf{F}^{\sim *}(0, v)\underline{h}^T\right] \quad \text{and} \\
&\mathbf{M}^*_{(0)}(v) = \mathbf{F}^{\sim *}(0, v).
\end{aligned}$$

To express the first moment, we use equation (17) and then, equation (20) to obtain

$$M_{(1)}(w) = \mathrm{LT}^{-1}\left[\underline{P}(0)\mathbf{M}^*_{(1)}(v)\underline{h}^T\right] = \mathrm{LT}^{-1}\left[\underline{P}(0)(\mathbf{R}v - \mathbf{A})^{-1}\mathbf{M}^*_{(0)}(v)\underline{h}^T\right],$$

which is by definition

$$\begin{aligned}
M^{(1)}(w) &= \mathrm{LT}^{-1}\left[\underline{P}(0)(\mathbf{R}v - \mathbf{A})^{-1}\mathbf{F}^{\sim *}(0, v)\underline{h}^T\right] \\
&= \mathrm{LT}^{-1}\left[\underline{P}(0)(\mathbf{R}v - \mathbf{A})^{-2}\mathbf{R}\underline{h}^T\right] \\
&= \mathrm{LT}^{-1}\left[\underline{P}(0)\frac{1}{v}(\mathbf{R}v - \mathbf{A})^{-1}\underline{h}^T\right],
\end{aligned}$$

since $(\mathbf{R}v - \mathbf{A})^{-2}\mathbf{R}\underline{h}^T = 1/v\ (\mathbf{R}v - \mathbf{A})^{-1}\underline{h}^T$, because $\mathbf{A}\underline{h}^T = \underline{0}^T$. The inverse transform gives the result of Theorem 2.

If the system is started from operational states, which is a rather realistic assumption, (i.e., $\forall i \in S^c$, $P_i(0) = 0$), then one can neglect the second term of the rhs of equation (17). This term stands for the time needed to start the reward accumulation (i.e., to enter S) when the system starts from S^c.

Another important analysis problem of MRMs is the probability distribution of the structure state process at completion, i.e., $P^c_{ij} = \Pr\{Z(C) = j \mid Z(0) = i\}$. For example, the required maintenance after a mission of a system can be estimated based on this performance measure. A closed form expression of the probability distribution at completion, by which its effective computation is possible, comes by the following theorem.

THEOREM 4. *The probability of being in state j at completion, given that the process started from state i, can be computed as follows:*

$$P^c_{ij} = \int_{w=0}^{\infty} \begin{bmatrix} e^{w\mathbf{R}_1^{-1}\beta} & \mathbf{0} \\ -\mathbf{A}_4^{-1}\mathbf{A}_3 e^{w\mathbf{R}_1^{-1}\beta} & \mathbf{0} \end{bmatrix}_{ij} dG(w). \tag{22}$$

PROOF. By the known transform domain measures, we have

$$\begin{aligned} P^c_{ij} &= \lim_{t\to\infty} \int_{w=0}^{\infty} F_{ij}(t,w)\,dG(w) = \lim_{s\to 0} \int_{w=0}^{\infty} F^{\sim}_{ij}(s,w)\,dG(w) \\ &= \int_{w=0}^{\infty} \left\{ \mathrm{LT}^{-1}[(v\mathbf{R}-\mathbf{A})^{-1}] \begin{bmatrix} \mathbf{R}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right\}_{ij} dG(w). \end{aligned} \tag{23}$$

From (23) and (21), the theorem comes. ∎

$P^c_{ij} = 0$ if $j \in S^c$, since the accumulated reward does not increase in S^c.

REFERENCES

1. R.A. Howard, *Dynamic Probabilistic Systems, Volume II: Semi-Markov and Decision Processes*, John Wiley and Sons, New York, (1971).
2. M. Telek, Some advanced reliability modelling techniques, Ph.D. Thesis, *Hungarian Academy of Science* (1994).
3. B.R. Iyer, L. Donatiello and P. Heidelberger, Analysis of performability for stochastic models of fault-tolerant systems, *IEEE Transactions on Computers* **C35**, 902–907 (1986).
4. K. Begain, L. Jereb, A. Puliafito and M. Telek, On-off Markov reward models, In *Proceedings of Relectronic'95*, Budapest, 95–100 (1995).
5. M.D. Beaudry, Performance-related reliability measures for computing systems, *IEEE Transactions on Computers* **C27**, 540–547 (1978).